ORTHOGONAL ALMOST COMPLEX STRUCTURES ON THE RIEMANNIAN PRODUCTS OF EVEN-DIMENSIONAL ROUND SPHERES

YUNHEE EUH AND KOUEI SEKIGAWA

Abstract. We discuss the integrability of orthogonal almost complex structures on Riemannian products of even-dimensional round spheres and give a partial answer to the question raised by E. Calabi concerning the existence of complex structures on a product manifold of a round 2-sphere and a round 4-sphere.
Mathematics Subsect Classification (2010) : 53C15, 53C21, 53C30
Keywords : orthogonal complex structure, even-dimensional sphere, curvature identity, Ricci $*$-tensor

1. Introduction

It is well-known that a $2n$-dimensional sphere S^{2n} admits an almost complex structure if and only if $n = 1$ or 3, and any almost complex structure on S^2 is integrable and also the complex structure on S^2 is unique with respect to the conformal structure on it. A 2-dimensional sphere S^2 equipped with this complex structure is biholomorphic to a complex projective line $\mathbb{C}P_1$. However, contrary to this, it is a long-standing open problem whether S^6 admits an integrable almost complex structure (namely, complex structure) or not. Lebrun [4] gave a partial answer to this problem, that is, proved that any orthogonal almost complex structure on a round 6-sphere is never integrable (see also [6], Corollary 5.2). On one hand, Sutherland proved that a connected product of even-dimensional spheres admits an almost complex structure if and only if it is a product of copies of S^2, S^6 and $S^2 \times S^4$ under more general setting ([7], Theorem 3.1). In [1], Calabi raised the problem whether the product manifold $V^2 \times S^4$ (V^2 is any closed, orientable surface) can admit an integrable almost complex structure or not. In the present note, we discuss the integrability of orthogonal almost complex structures on a Riemannian product of round 2-spheres, 6-spheres and Riemannian product manifolds of a round 2-sphere and a round 4-sphere, and prove the following.

Theorem A. *An orthogonal almost complex structure on a Riemannian product of round 2-spheres, round 6-spheres, and Riemannian product manifolds of a round 2-sphere and round a 4-sphere is integrable if and only if it is the product of the canonical complex structures on round 2-spheres.*

Remark 1. *Let M be any Riemannian product of round 2-spheres. Then, the product of the canonical complex structures of the round 2-spheres is necessarily an orthogonal complex structure on M.*

From Theorem A, we have the following partial answer to the above mentioned problem by Calabi.

Corollary B. *Any orthogonal almost complex structure on a Riemannain product of a round 2-sphere and a round 4-sphere is never integrable.*

Remark 2. *An explicit example of an orthogonal almost Hermitian structure on a Riemannian product of a round 2-sphere and a round 4-sphere was introduced and its geometric property was discussed in* [3].

We denote by $S^m(\kappa)$ an m-dimensional round sphere of positive constant sectional curvature κ. Throughout the present paper, we shall mean by *a round m-sphere* an oriented m-dimensional sphere with constant sectional curvature.

The authors would like to express their thanks to Professor H. Hashimoto for drawing their attention to the present topic of this paper and also to the referee for his valuable suggestions.

2. Preliminaries

Let $M = (M, J, <,>)$ be a $2n$-dimensional almost Hermitian manifold. We denote by ∇ the Levi-Civita connection and R the curvature tensor of M defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}Z \tag{2.1}$$

for X, Y, $Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. We denote the Ricci $*$-tensor of M by ρ^* which is defined by

$$\begin{aligned} \rho^*(X,Y) =& \mathrm{tr}\big(Z \longmapsto R(X,JZ)JY\big) \\ =& \frac{1}{2}\mathrm{tr}\big(Z \longmapsto R(X,JY)JZ\big) \end{aligned} \tag{2.2}$$

for X, Y, $Z \in \mathfrak{X}(M)$. We here note that the Ricci $*$-tensor ρ^* satisfies the following equality

$$\rho^*(X,Y) = \rho^*(JY,JX) \tag{2.3}$$

for X, $Y \in \mathfrak{X}(M)$. Thus from (2.3), we see that ρ^* is symmetric if and only if ρ^* is J-invariant. We also denote by N the Nijenhuis tensor of the almost complex structure J defined by

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] \tag{2.4}$$

for X, $Y \in \mathfrak{X}(M)$. It follows from the celebrated theorem of Newlander and Nirenberg [5] that the almost complex structure J is integrable if and only if $N = 0$ holds everywhere on M. An almost Hermitian manifold with an integrable almost complex structure is called a Hermitian manifold.

Now, we set

$$R(X,Y,Z,W) =< R(X,Y)Z, W > \tag{2.5}$$

for X, Y, Z, $W \in \mathfrak{X}(M)$. Gray [2] proved the following result which plays an important role in our forthcoming arguments of the present paper.

Theorem 2.1. *The curvature tensor R of a Hermitian manifold $M = (M, J, <,>)$ satisfies the following identity:*

$$\begin{aligned} &R_{WXYZ} + R_{JWJXJYJZ} - R_{JWJXYZ} - R_{JWXJYZ} \\ &\quad - R_{JWXYJZ} - R_{WJXJYZ} - R_{WJXYJZ} - R_{WXJYJZ} = 0 \end{aligned}$$

for any W, X, Y, $Z \in \mathfrak{X}(M)$.

3. Lammas

We shall prepare several lemmas prior to the proof of Theorem A. First of all, we note that orthogonal almost complex structures on the Riemannian products of even-dimensional round spheres do not depend on the order of the factors. Now, we consider the Riemannian product $M = S^2(\alpha) \times M'$, where M' is a Riemannian product of round 2-spheres, round 6-spheres and Riemannian product manifolds of a round 2-sphere and a round 4-sphere.

Lemma 3.1. *Let J be an orthogonal complex structure on M. Then, J induces a canonical complex structure on $S^2(\alpha)$ and an orthogonal almost complex structure on $\{p_1\} \times M'$ for each point $p_1 \in S^2(\alpha)$.*

Proof of Lemma 3.1. We denote by π_1 and π_2 the canonical projections defined by $\pi_1 : M \longrightarrow S^2(\alpha)$ and $\pi_2 : M \longrightarrow M'$, respectively. We set

$$x_1 = d\pi_1(x), \quad x_2 = d\pi_2(x) \tag{3.1}$$

for any $x \in T_pM$, $p = (p_1, p_2) \in S^2(\alpha) \times M'$. The tangent space T_pM is identified with the orthogonal direct sum of $T_{p_1}S^2(\alpha)$ and $T_{p_2}M'$ in the natural way. Let x, $y \in T_{p_1}S^2(\alpha)$ with $x \perp y$, $|x| = |y| = 1$. Then, we get

$$R(x, y, x, y) = -\alpha. \tag{3.2}$$

Here, since $dim\ S^2(\alpha) = 2$, we may set

$$(Jx)_1 =< Jx, y > y, \qquad (Jy)_1 =< Jy, x > x. \tag{3.3}$$

Now, taking account of (3.3), we get further

$$\begin{aligned} &R(Jx, Jy, Jx, Jy) \\ &= R((Jx)_1 + (Jx)_2, (Jy)_1 + (Jy)_2, (Jy)_1 + (Jy)_2, (Jx)_1 + (Jx)_2) \\ &= R((Jx)_1, (Jy)_1, (Jx)_1, (Jy)_1) + R((Jx)_2, (Jy)_2, (Jx)_2, (Jy)_2) \\ &= -\alpha\big(|(Jx)_1|^2|(Jy)_1|^2 - < (Jx)_1, (Jy)_1 >^2\big) \\ &\quad + R((Jx)_2, (Jy)_2, (Jx)_2, (Jy)_2) \\ &= -\alpha|(Jx)_1|^2|(Jy)_1|^2 + R_2((Jx)_2, (Jy)_2, (Jx)_2, (Jy)_2), \end{aligned} \tag{3.4}$$

where R_2 is the curvature tensor of M'.

$$\begin{aligned} R(Jx, Jy, x, y) =& R((Jx)_1, (Jy)_1, x, y) \\ =& < Jx, y >< x, Jy > R(y, x, x, y) \\ =& \alpha < Jx, y >< x, Jy > \\ =& -\alpha < x, Jy >^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned} R(Jx, y, Jx, y) =& R((Jx)_1, y, (Jx)_1, y) \\ =& < Jx, y >^2 R(y, y, y, y) \\ =& 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} R(Jx, y, x, Jy) =& R((Jx)_1, y, x, (Jy)_1) \\ =& - < Jx, y >^2 R(y, y, x, x) \\ =& 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} R(x, Jy, x, Jy) =& R(x, (Jy)_1, x, (Jy)_1) \\ =& < Jy, x >^2 R(x, x, x, x) \\ =& 0. \end{aligned} \tag{3.8}$$

Thus, from Theorem 2.1 and (3.2)∼(3.8), we have

$$\begin{aligned}0 &= R(x,y,x,y) + R(Jx,Jy,Jx,Jy) - 2R(Jx,Jy,x,y) \\ &\quad - R(Jx,yJx,y) - 2R(Jx,y,x,Jy) - R(x,Jy,x,Jy) \\ &= -\alpha\{1 - |(Jx)_1|^2|(Jy)_1|^2\} + R_2((Jx)_2,(Jy)_2,(Jx)_2,(Jy)_2).\end{aligned} \tag{3.9}$$

Since M' is non-negatively curved, we see that

$$R_2((Jx)_2,(Jy)_2,(Jx)_2,(Jy)_2) \le 0 \tag{3.10}$$

for all x, $y \in T_{p_1}S^2(\alpha)$. Thus, from (3.9) and (3.1), we see that

$$|(Jx)_1| = 1 \quad \text{and} \quad |(Jy)_1| = 1 \tag{3.11}$$

and hence, $Jx \in T_{p_1}S^2(\alpha)$ and $Jy \in T_{p_1}S^2(\alpha)$ for any orthogonal pair $\{x, y\}$ in $T_{p_1}S^2(\alpha)$. Since $d\pi_1$ is a linear map from T_pM onto $T_{p_1}S^2(\alpha)$, from (3.11), we may easily see that $Jx \in T_{p_1}S^2(\alpha)$ for all $x \in T_{p_1}S^2(\alpha)$, and hence $J(T_{p_1}S^2(\alpha)) = T_{p_1}S^2(\alpha)$. Therefore we see also that $J(T_{p_2}M') = T_{p_2}M'$.

Now, for each $p_1 \in S^2(\alpha)$, we denote by $J' = J'(p_1)$ the induced almost complex structure on $\{p_1\} \times M'$ in the above Lemma 3.1. Then we have the following.

Lemma 3.2. *The almost complex structure J' is integrable (and hence defines a complex structure on $\{p_1\} \times M'$).*

Proof of Lemma 3.2. Let N' be the Nijenhuis tensor of the almost complex structure J'. Then, taking account of Lemma 3.1, we have

$$\begin{aligned}N'(X',Y') &= [J'X',J'Y'] - [X',Y'] - J'[J'X',Y'] - J'[X',J'Y'] \\ &= [JX',JY'] - [X',Y'] - J'[JX',Y'] - J'[X',JY'] \\ &= [JX',JY'] - [X',Y'] - J[JX',Y'] - J[X',JY'] \\ &= N(X',Y') \\ &= 0\end{aligned} \tag{3.12}$$

for all X', $Y' \in \mathfrak{X}(M')$. Therefore, from (3.12), we see that the induced almost complex structure J' on $\{p_1\} \times M'$ is integrable for each $p_1 \in S^2(\alpha)$. □

From Lemmas 3.1 and 3.2, if M' involves a round 2-sphere as a factor, by a suitable reordering of the factors, we may assume that M is expressed in the form $M' = S^2(\alpha) \times M''$, where M'' expressed by the similar form as M'. Then, applying Lemma 3.2 to M', it follows that the orthogonal complex structure J' induces a complex structure on M''. Repeating the similar operations, we may assume that M is expressed in the form $M = M_1 \times M_2$, where $M_1 = S_1^2(\alpha_1) \times \cdots \times S_s^2(\alpha_s)$ $(0 \le \alpha_1 \le \cdots \le \alpha_s)$ and M_2 does not involve a round 2-sphere, and further that the orthogonal almost complex structure J on M induces a canonical orthogonal complex structure on $M_1 \times \{p_2\}$ for each point $p_2 \in M_2$ and an orthogonal almost complex structure on $\{p_1\} \times M_2$ for each point $p_1 \in M_1$, respectively. Thus, taking account of the result due to Sutherland ([7], Theorem 3.1), we have the following.

Lemma 3.3. *Let M be a Riemannian product of round 2-spheres, round 6-spheres and Riemannian product manifolds of a round 2-sphere and a round 4-sphere, and J be an orthogonal complex structure on M. Then, M takes of the form $M = M' \times M''$ (after suitable reordering of the factors), where M' (resp. M'') is a Riemannian product of round 2-spheres (resp. a Riemannian product of round 6-spheres), and further, J induces a canonical orthogonal complex structure on $M' \times \{p''\}$ for each point $p'' \in M''$ and an orthogonal complex structure on $\{p'\} \times M''$ for each point $p' \in M'$, respectively.*

Now, we shall show the following.

Lemma 3.4. *Let $M = (M, <, >)$ be the Riemannian product of round 6-spheres $S^6_a(\beta_a) = (S^6, <, >_a)$ $(0 < \beta_1 \leq \beta_2 \cdots \leq \beta_t, a = 1, 2, ..., t)$, and J be an orthogonal almost complex structure on M. Then, for each point $(p_1, \cdots, p_{a-1}, p_{a+1}, \cdots, p_t) \in S^6_1(\beta_1) \times \cdots \times S^6_{a-1}(\beta_{a-1}) \times S^6_{a+1}(\beta_{a+1}) \times \cdots \times S^6_t(\beta_t)$, J induces an orthogonal almost complex structure on $\{(p_1, \cdots, p_{a-1}, p_{a+1}, \cdots, p_t)\} \times S^6_a(\beta_a)$.*

Proof of Lemma 3.4. Let $p = (p_1, p_2, \cdots, p_t) \in M$ $(p_a \in S^6_a(\beta_a),\ a = 1, 2, \cdots, t)$ be any point of M and $\{e(a)_i\}$ $(i = 1, 2, \cdots, 6)$ be any orthonormal basis of $T_{p_a}S^6_a(\beta_a)$. We denote by $R_{(a)}$ the curvature tensor of $S^6_a(\beta_a)$. Then, we have

$$R(x, y)z = R_{(a)}(x, y)z, \tag{3.13}$$

and

$$R_{(a)}(x, y)z = \beta_a\big(< y, z >_a x - < x, z >_a y\big) \tag{3.14}$$

for $x,\ y,\ z \in T_{p_a}S^6_a(\beta_a)$. Now, we set

$$Je(a)_i = \sum_{c=1}^{t}\big(\sum_{j=1}^{6} J(a, c)_{ij} e(c)_j\big) \tag{3.15}$$

for $1 \leq i \leq 6$ and $1 \leq a \leq t$. Then since $< Je(a)_i, e(b)_j >= - < e(a)_i, Je(b)_j >$, from (3.15), we have

$$\begin{aligned} < Je(a)_i, e(b)_j > &=< \sum_c \sum_k J(a, c)_{ik} e(c)_k, e(b)_j > \\ &= \sum_c \sum_k J(a, c)_{ik} \delta_{cb} \delta_{kj} \\ &= J(a, b)_{ij} \end{aligned}$$

and

$$\begin{aligned} < e(a)_i, Je(b)_j > &=< e(a)_i, \sum_c \sum_k J(b, c)_{jk} e(c)_k > \\ &= \sum_c \sum_k J(b, c)_{jk} \delta_{ac} \delta_{ik} \\ &= J(b, a)_{ji}, \end{aligned}$$

and hence, we have

$$J(a, b)_{ij} = -J(b, a)_{ji} \tag{3.16}$$

for $1 \leq a, b \leq t$ and $1 \leq i, j \leq 6$. On one hand, since $J^2 = -id$, from (3.15), we have

$$\begin{aligned} -e(a)_i &= J(Je(a)_i) \\ &= J\big(\sum_c \sum_j J(a, c)_{ij} e(c)_j\big) \\ &= \sum_c \sum_d \sum_{j,k} J(a, c)_{ij} J(c, d)_{jk} e(d)_k, \end{aligned}$$

and hence,

$$\sum_c \sum_j J(a, c)_{ij} J(c, d)_{jk} = -\delta_{ik}\delta_{ad} \tag{3.17}$$

for $1 \leq i, k \leq 6$ and $1 \leq a, d \leq t$. Here, we shall calculate the components of the Ricci $*$-tensor ρ^*. From (3.13), (3.15), (3.16) and (3.17), we have

$$\begin{aligned}
&\rho^*(e(a)_i, e(a)_j)\\
&= -\frac{1}{2}\sum_c\sum_k R(e(a)_i, Je(a)_j, e(c)_k, Je(c)_k)\\
&= -\frac{1}{2}\sum_k R(e(a)_i, Je(a)_j, e(a)_k, Je(a)_k)\\
&= -\frac{1}{2}\sum_k R_{(a)}\big(e(a)_i, \sum_l J(a,a)_{jl}e(a)_l, e(a)_k, \sum_u J(a,a)_{ku}e(a)_u\big)\\
&= -\frac{1}{2}\sum_{k,l,u} J(a,a)_{jl}J(a,a)_{ku}R_{(a)}(e(a)_i, e(a)_l, e(a)_k, e(a)_u)\\
&= -\frac{\beta_a}{2}\sum_{k,l,u} J(a,a)_{jl}J(a,a)_{ku}\{\delta_{lk}\delta_{iu} - \delta_{ik}\delta_{lu}\}\\
&= -\frac{\beta_a}{2}\{-\delta_{ji} - \delta_{ji}\} = \beta_a\delta_{ij},
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\rho^*(e(a)_i, e(b)_j)\\
&= -\frac{1}{2}\sum_c\sum_k R(e(a)_i, Je(b)_j, e(c)_k, Je(c)_k)\\
&= -\frac{1}{2}\sum_k R(e(a)_i, \sum_l J(b,a)_{jl}e(a)_l, e(a)_k, \sum_u J(a,a)_{ku}e(a)_u)\\
&= -\frac{1}{2}\sum_{k,l,u} J(b,a)_{jl}J(a,a)_{ku}R_{(a)}(e(a)_i, e(a)_l, e(a)_k, e(a)_u)\\
&= -\frac{\beta_a}{2}\sum_{k,l,u} J(b,a)_{jl}J(a,a)_{ku}\{\delta_{lk}\delta_{iu} - \delta_{ik}\delta_{lu}\}\\
&= -\frac{\beta_a}{2}\{J(b,a)_{jk}J(a,a)_{ki} - \sum_l J(b,a)_{jl}J(a,a)_{il}\}\\
&= -\frac{\beta_a}{2}\{-\delta_{ji}\delta_{ba} - \delta_{ji}\delta_{ba}\} = \beta_a\delta_{ij}\delta_{ab},
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
&\rho^*(e(a)_i, Je(a)_j)\\
&= \frac{1}{2}\sum_c\sum_k R(e(a)_i, e(a)_j, e(c)_k, Je(c)_k)\\
&= \frac{1}{2}\sum_k R(e(a)_i, e(a)_j, e(a)_k, Je(a)_k)\\
&= \frac{1}{2}\sum_{k,l} J(a,a)_{kl}R_{(a)}(e(a)_i, e(a)_j, e(a)_k, e(a)_l)\\
&= \frac{\beta_a}{2}\sum_{k,l} J(a,a)_{kl}\{\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}\}\\
&= \frac{\beta_a}{2}\{J(a,a)_{ji} - J(a,a)_{ij}\}\\
&= \beta_a J(a,a)_{ji},
\end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&\rho^*(e(a)_i, Je(b)_j)\\
&\quad = \frac{1}{2}\sum_c\sum_k R(e(a)_i, e(b)_j, e(c)_k, Je(c)_k)\\
&\quad = \frac{1}{2}\sum_{c,d}\sum_{k,l} J(c,d)_{kl} R(e(a)_i, e(b)_j, e(c)_k, e(d)_l)\\
&\quad = -\beta_a\delta_{ab}J(a,b)_{ij},
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&\rho^*(Je(a)_i, e(a)_j)\\
&\quad = -\frac{1}{2}\sum_c\sum_k R(Je(a)_i, Je(a)_j, e(c)_k, Je(c)_k)\\
&\quad = -\frac{1}{2}\sum_c\sum_{k,l,u,v} J(a,c)_{il}J(a,c)_{ju}J(c,c)_{kv}R_{(c)}(e(c)_l, e(c)_u, e(c)_k, e(c)_v)\\
&\quad = -\frac{1}{2}\sum_c\beta_c\sum_{k,l,u,v} J(a,c)_{il}J(a,c)_{ju}J(c,c)_{kv}\{\delta_{uk}\delta_{lv}-\delta_{lk}\delta_{uv}\}\\
&\quad = -\frac{1}{2}\sum_c\beta_c\{\sum_{k,l} J(a,c)_{il}J(a,c)_{jk}J(c,c)_{kl}\\
&\qquad\qquad -\sum_{k,u} J(a,c)_{ik}J(a,c)_{ju}J(c,c)_{ku}\}\\
&\quad = -\frac{1}{2}\sum_c\beta_c\{-\sum_l J(a,c)_{il}\delta_{jl}\delta_{ac}+\sum_u J(a,c)_{ju}\delta_{iu}\delta_{ac}\}\\
&\quad = \frac{1}{2}\beta_a J(a,a)_{ij}-\frac{1}{2}\beta_a J(a,a)_{ji}\\
&\quad = \beta_a J(a,a)_{ij},
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
&\rho^*(Je(b)_i, e(a)_j)\\
&\quad = -\frac{1}{2}\sum_c\sum_k R(Je(b)_i, Je(a)_j, e(c)_k, Je(c)_k)\\
&\quad = -\frac{1}{2}\sum_c\sum_{k,l,u,v} J(b,c)_{il}J(a,c)_{ju}J(c,c)_{kv}R_{(c)}(e(c)_l, e(c)_u, e(c)_k, e(c)_v)\\
&\quad = -\frac{1}{2}\sum_c\beta_c\sum_{k,l,u,v} J(b,c)_{il}J(a,c)_{ju}J(c,c)_{kv}\{\delta_{uk}\delta_{lv}-\delta_{lk}\delta_{uv}\}\\
&\quad = -\frac{1}{2}\sum_c\beta_c\{\sum_{k,l} J(b,c)_{il}J(a,c)_{jk}J(c,c)_{kl}\\
&\qquad\qquad -\sum_{k,u} J(b,c)_{ik}J(a,c)_{ju}J(c,c)_{ku}\}\\
&\quad = -\frac{1}{2}\sum_c\beta_c\{-\sum_l \delta_{jl}\delta_{ac}J(b,c)_{il}+\sum_u \delta_{iu}\delta_{bc}J(a,c)_{ju}\}\\
&\quad = \frac{1}{2}\beta_a J(b,a)_{ij}-\frac{1}{2}\beta_a J(a,b)_{ji}\\
&\quad = -\beta_a J(a,b)_{ij}.
\end{aligned}
\tag{3.23}
$$

Thus, from (3.18) and (3.19), we see that ρ^* is symmetric (and hence, J-invariant). Further, from (3.21) and (3.23), taking account of the symmetry of ρ^*, we have $J(a,b)_{ij} = 0$ for $a \neq b$, and hence

$$J(T_{p_a} S^6_a(\beta_a)) = T_{p_a} S^6_a(\beta_a), \quad a = 1, 2, \cdots, t. \tag{3.24}$$

Therefore, from (3.24), we see that J induce an almost complex structure on $\{(p_1, \cdots, p_{a-1}, p_{a+1}, \cdots, p_t)\} \times S^6_a(\beta_a)$ for each $(p_1, \cdots, p_{a-1}, p_{a+1}, \cdots, p_t) \in S^6_1(\beta_1) \times \cdots \times S^6_{a-1}(\beta_{a-1}) \times S^6_{a+1}(\beta_{a+1}) \times \cdots \times S^6_t(\beta_t)$. □

Lemma 3.5. *Any orthogonal almost complex structure on a Riemannian product of round 6-spheres is never integrable.*

Proof of Lemma 3.5. Let $M = (M, <, >)$ be a Riemannian product of round 6-spheres $S^6_a(\beta_a)$ $(a = 1, 2, \cdots, t)$, and assume that M admits an orthogonal complex structure denoted by J. Then, taking account of the results in [4], it suffices to consider the case where $t \geq 2$. From Lemma 3.4, for each point $(p_1, \cdots, p_{t-1}) \in S^6_1(\beta_1) \times \cdots \times S^6_{t-1}(\beta_{t-1})$, J induces an orthogonal almost complex structure on $\{(p_1, \cdots, p_{t-1})\} \times S^6_t(\beta_t)$. Then, we may show that the induced orthogonal almost complex structure is integrable by slightly modifying the proof of Lemmas 3.1 and 3.2. But, this is a contradiction. □

4. Proof of Theorem A

In this section, we prove Theorem A based on the arguments in §3. Let $M = (M, <, >)$ be a Riemannian product of round 2-spheres, round 6-spheres, and Riemannian product manifolds of a round 2-sphere and a round 4-sphere, assume that M admits an orthogonal complex structure. We denote it by J. Then, from Lemma 3.3, we see that M is of the form $M = M' \times M''$, where M' is of the form $M' = S^2(\alpha_1) \times \cdots \times S^2(\alpha_s)$ and M'' is of the form $M'' = S^6(\beta_1) \times \cdots \times S^6(\beta_t)$, respectively, and further, J induces an orthogonal complex structure on $\{p'\} \times M''$ for each point $p' \in M'$. Therefore, from Lemmas 3.1 and 3.2 and the uniqueness of the canonical complex structure on a round 2-sphere, J is an orthogonal complex structure on M. Therefore, taking account of Lemma 3.1 we see that J is a product of the canonical complex structures on these round 2-spheres. The converse is evident by Remark 1. This completes the proof of Theorem A.

References

[1] E. Calabi, *Construction and properties of some 6-dimensional almost complex manifold*, Trans. Amer. Math. Soc. **87** (1958), 407–438.

[2] A. Gray, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J. **28** (1976), 601–612.

[3] H. Hashimoto, T. Koda, K. Mashimi and K. Sekigawa, *Extrinsic homogeneous almost Hermitian 6-dimensional submanifolds in the Octonions*, Kodai Math. J. **30** (2007), 297-321.

[4] C. Lebrun, *Orthogonal complex structures on S^6*, Proc. Amer. Math. Soc. **101** (1987), 136–138.

[5] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. **65** (1957), 391–404.

[6] K. Sekigawa and L. Vanhecke, *Almost Hermitian manifolds with vanishing first Chern class or Chern numbers*, Rend. Sem. Mat. Univ. Politec. Torino **50** (1992), 195–208.

[7] W. A. Sutherland, *A note on almost complex and weakly complex structures*, J. of London Math. Soc. **40** (1965), 705–712.

Y. Euh : Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
E-mail address: prettyfish@skku.edu

K. Sekigawa : Department of Mathematics, Niigata University, Niigata 950-2181, Japan
E-mail address: sekigawa@math.sc.niigata-u.ac.jp